Poisson integrator for symmetric rigid bodies

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September 2004

Abstract

We derive an explicit second order reversible Poisson integrator for symmetric rigid bodies in space (i.e. without a fixed point). The integrator is obtained by applying a splitting method to the Hamiltonian after reduction by the S^1 body symmetry. In the particular case of a magnetic top in an axisymmetric magnetic field (i.e. the Levitron) this integrator preserves the two momentum integrals. The method is used to calculate the complicated boundary of stability near a linearly stable relative equilibrium of the Levitron with indefinite Hamiltonian.

1 Introduction

The Levitron is a magnetic spinning top that can levitate in the air repelled by a base magnet. It is an interesting example of a symmetric rigid body (without a fixed point). The Levitron was invented by Roy Harrigan [7], see [12] for the interesting history of its invention. A model obtained from adiabatic approximations has been given by Berry [2]. The first derivation of the exact Hamiltonian equations of motion of the Levitron with six degrees of freedom was given in [3, 4]. There we have shown that the relative equilibrium of the top spinning aligned with the z-axis can be linearly stable when the spin rate is in some interval. Even if this relative equilibrium is elliptic it is not known whether it is Lyapunov stable. This is a hard problem because the reduced Hamiltonian is not definite at the relative equilibrium, as is typical for gyroscopically stabilized systems. To numerically study the stability near this equilibrium therefore calls for a good geometric integrator, and thus motivated this paper.

While the calculation in [3] was done in local coordinates adapated to the relative equilibrium, in [4] a global reduction of the body symmetry was done. This leads to nice reduced equations with a Poisson structure and two Casimirs in $\mathbb{R}^{12}$. The variables in this description are the natural variables of the problem: the centre of mass $\mathbf{r}$, the conjugate momentum $\mathbf{p}$, the spatial angular momentum $\mathbf{l}$ and the axis of the top $\mathbf{a}$. As we show in the present paper this Poisson structure is not specific to the Levitron, but appears in general when a symmetric rigid body (with or without fixed point) is reduced by the body symmetry. Similarly system of interacting rigid bodies can be treated in the same way. The structure of these reduced equations is idealy suited for the derivation of a Poisson integrator.

Geometric integrators are numerical methods for the solution of differential equations that preserve geometric structure of a problem, see [11, 8]. In our application this will be the Poisson structure of the reduced equations and the reversibility of the vector field. Poisson integrators are known to approximately conserve the energy for long times [8]. This is so because an (approximate) Poisson integrator can be shown to be an exact integrator for a modified Hamiltonian [8]. But this modification is exponentially small, hence the good long term stability. What is maybe even more important is that our Poisson integrator will exactly preserve not only the Poisson structure but also the two known conserved momenta (one of which is turned into a Casimir by reduction).

General purpose Poisson integrators are often implicit. But if one is willing to derive a method specifically for one problem it is easy to get explicit methods. One popular way of doing this is to use so called splitting methods [9, 8]. The idea is to decompose the Hamiltonian into a sum, so that the terms depend only on subsets of the variables. A typical example is that of kinetic and potential energy, assuming that the latter depends on positions, and the former on momenta only. We will use exactly this approach. Let us remark that the geometric formulation of the equations of the reduced system is crucial for this step. E.g. in the Euler-angle like variables employed in [3] a splitting is impossible. However, after the right geometrically nice equations of motion have been obtained by reduction, the derivation of a geometric integrator is straightforward. Other methods for the symplectic integration of (systems of) rigid bodies have been given in [10, 1, 6]. Our method is much more special (special to *symmetric* rigid bodies), but as a result the integration scheme is also simpler.

The paper is organised as follows. We derive the Poisson structure and the reduced equations of motion for symmetric rigid bodies in the next section. Then we derive a second order reversible Poissson integrator. For systems with additional S^1 or $E(3)$ symmetry the integrator is also momentum preserving. In the final sections we specialise to the Levitron, and apply the integrator to the computation of the fractal boundaries of the region of stability.

2 Reduced Equations of Motion

A symmetric rigid body is a simple mechanical system with symmetry. Its configuration space is $\mathbb{R}^3 \times SO(3)$, where the position of the centre of mass is $\mathbf{r} \in \mathbb{R}^3$ and the orientation of the body in space is given by an orthogonal matrix $\mathbf{R} \in SO(3)$. The body is assumed to have two equal moments of inertia $\Theta_1 = \Theta_2$. Hence the potential is invariant under right multiplication by the one parameter symmetry group S^1 of rotations $\mathbf{R}_3(\varphi)$ which leaves the third axis in the body frame $\mathbf{e}_3$ fixed. As a result the potential only depends on $\mathbf{a} = \mathbf{R}\mathbf{e}_3 = \mathbf{R}\mathbf{R}_3(\varphi)\mathbf{e}_3$, the coordinates of the symmetry axis of the top, and not on all of $SO(3)$. Denote by $\mathbf{l}$ the angular momentum of the body in the space fixed frame. By Noethers theorem the conserved momentum corresponding to S^1 is the third component L_3 of the momentum in the body fixed frame, in space fixed variables this is $L_3 = \langle \mathbf{l}, \mathbf{a} \rangle$, where $\langle , \rangle$ is the standard Euclidean scalar product. The reduction by the S^1-symmetry leads to a system on $T^*(\mathbb{R}^3 \times S^2)$. This reduced phase space is realised as the leaf of a Poisson structure on $\mathbb{R}^{12}$ with two Casimirs $C_1 = \langle \mathbf{a}, \mathbf{a} \rangle$, $C_2 = \langle \mathbf{l}, \mathbf{a} \rangle$. For the components of $\mathbf{r}$ and $\mathbf{p}$ the standard symplectic structure holds, $\{p_i, r_j\} = \delta_{ij}$. For $(\mathbf{a}, \mathbf{l})$ the Euler-Poisson structure (with a minus sign, because we are working in the space fixed frame) holds, $\{a_i, l_j\} = -\epsilon_{ijk} a_k$, $\{l_i, l_j\} = -\epsilon_{ijk} l_k$. All other fundamental Poisson brackets are identically equal to zero. The brackets between the components of $\mathbf{p}$ and $\mathbf{r}$ are standard. The brackets between components of $\mathbf{a}$ and $\mathbf{l}$ can be derived by a tedious but simple computation using local coordinates in $SO(3)$, e.g. the Euler angles.

The reduced Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m} + \frac{\mathbf{l}^2}{2\Theta_1} + U(\mathbf{r}, \mathbf{a}) + c, \quad c = \frac{1}{2}\left(\frac{1}{\Theta_3} - \frac{1}{\Theta_1}\right) \langle \mathbf{l}, \mathbf{a} \rangle^2 . \tag{1}$$

The constant c can be ignored in the following. This Hamiltonian is written in terms of coordinates in the space fixed frame. In general in rigid body dynamics the body fixed frame is used because in this comoving frame the kinetic energy only depends on the angular momenta, but not on the location $\mathbf{R} \in SO(3)$. However, for a symmetric rigid body with diagonalised tensor of inertia $\boldsymbol{\Theta} = \operatorname{diag}(\Theta_1, \Theta_1, \Theta_3)$ the following little computation shows that the kinetic energy is a function only of the angular momenta even when expressed in the space fixed frame. Denote by $\mathbf{L} = \mathbf{R^t l}$ the angular momentum of the body in the body fixed frame. Then the kinetic energy is

$$
\begin{aligned}
T &= \frac{1}{2}\mathbf{L}^t\boldsymbol{\Theta}^{-1}\mathbf{L} = \frac{1}{2}\mathbf{l}^t\mathbf{R}\boldsymbol{\Theta}^{-1}\mathbf{R}^t\mathbf{l} \\
&= \frac{1}{2}\mathbf{l}^t\mathbf{R}\left(\frac{1}{\Theta_1}I + \left(\frac{1}{\Theta_3} - \frac{1}{\Theta_1}\right)\mathbf{e}_z\mathbf{e}_z^t\right)\mathbf{R}^t\mathbf{l} \\
&= \frac{1}{2\Theta_1}\mathbf{l}^2 + \frac{1}{2}\left(\frac{1}{\Theta_3} - \frac{1}{\Theta_1}\right)\langle\mathbf{a},\mathbf{l}\rangle^2.
\end{aligned}
$$

As already noted for a symmetric body the angular momentum corresponding to the rotation about the symmetry axis of the body $L_3 = \langle\mathbf{a},\mathbf{l}\rangle$ is a constant of motion, and hence the constant c in (1) merely changes the value of the Hamiltonian.

The corresponding equations of motion for the general Hamiltonian are

$$
\begin{aligned}
\dot{\mathbf{a}} &= \frac{\partial H}{\partial \mathbf{l}} \times \mathbf{a}, \quad & \dot{\mathbf{l}} &= \frac{\partial H}{\partial \mathbf{l}} \times \mathbf{l} + \frac{\partial H}{\partial \mathbf{a}} \times \mathbf{a}, \\
\dot{\mathbf{r}} &= \frac{\partial H}{\partial \mathbf{p}}, & \dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{r}}\,.
\end{aligned}
\tag{2}
$$

Up to a sign the first pair of equations looks like the well known Euler-Poisson (or Kirchhoff) equations for the rigid body. The crucial difference is that those equations are written in a frame moving with the body, while our equations are written in the space fixed frame. For the particular Hamiltonian (1) of a symmetric rigid body the equations of motion simplify slightly to

$$
\begin{aligned}
\dot{\mathbf{a}} &= \frac{1}{\Theta_1}\mathbf{l} \times \mathbf{a}, \quad & \dot{\mathbf{l}} &= \nabla_a U \times \mathbf{a}, \\
\dot{\mathbf{r}} &= \frac{1}{m}\mathbf{p}, & \dot{\mathbf{p}} &= -\nabla_r U\,.
\end{aligned}
\tag{3}
$$

The energy H and the Casimirs $C_1 = \langle\mathbf{a},\mathbf{a}\rangle$ and $C_2 = \langle\mathbf{a},\mathbf{l}\rangle = L_3$ are the obvious conserved quantities. The original equations are reversible under flipping the sign of the momenta. After reduction the linear reversing map $\rho : (\mathbf{r},\mathbf{a},\mathbf{p},\mathbf{l}) \mapsto (\mathbf{r},\mathbf{a},-\mathbf{p},-\mathbf{l})$ has the property that it changes the sign of the Casimir C_2. Nevertheless the equations (3) do satisfy $\rho \circ X = -X \circ \rho$, where X denotes the 12 components of the vector field (3). Even though ρ in general maps to a different symplectic leave the vector field is reversible in the standard sense, and therefore it is advantageous to use a reversible integration method. The equations (2) are not in general ρ-reversible, e.g. when $\nabla_l H \not\parallel \mathbf{l}$.

In molecular dynamics a typical system is that of N interacting rigid bodies, the configuration space is $(\mathbb{R}^3 \times SO(3))^N$. Assuming that each rigid body is symmetric the symmetry group is $\mathbb{T}^N$ and a similar reduction can be performed. The symmetry assumption implies that the potential depends on the positions $\mathbf{r}_i$ and the axes $\mathbf{a}_i$ only. The reduced system is described by a Poisson structure on $\mathbb{R}^{6N}$ with Casimirs $\langle \mathbf{a}_i, \mathbf{a}_i \rangle$ and $\langle \mathbf{a}_i, \mathbf{l}_i \rangle$, $i = 1, \ldots, N$ whose symplectic leaf is $(T^*(\mathbb{R}^3 \times S^2))^N$. The resulting equations of motion are like (2) where each vector $\mathbf{r}, \mathbf{p}, \mathbf{a}, \mathbf{l}$ carries an index $i = 1, \ldots, N$.

3 Geometric Integrator

The integrator is constructed using a splitting method, in which T and U are separately taken as generators of flows. The procedure is well known, see e.g. [8, 9], and rests on the fact that the kinetic energy is independent of the momenta and the potential energy is independent of the coordinates. However, since we are working in a non-constant Poisson structure it is not immediately clear that the splitting actually works as it does in the symplectic structure. It turns out that it does. The equations for the split flows are obtained by replacing H by $T(\mathbf{p}, \mathbf{l})$ in (2), which assuming $\mathbf{l} \parallel \nabla_l T$ gives

$$\begin{aligned} \frac{d}{dt_T}\mathbf{a} &= \frac{1}{\Theta_1}\mathbf{l} \times \mathbf{a}, & \frac{d}{dt_T}\mathbf{l} &= 0, \\ \frac{d}{dt_T}\mathbf{r} &= \frac{1}{m}\mathbf{p}, & \frac{d}{dt_T}\mathbf{p} &= 0\,, \end{aligned} \tag{4}$$

and by replacing H by $U(\mathbf{r}, \mathbf{a})$ in (2), which gives

$$\begin{aligned} \frac{d}{dt_U}\mathbf{a} &= 0, & \frac{d}{dt_U}\mathbf{l} &= \nabla_a U \times \mathbf{a}, \\ \frac{d}{dt_U}\mathbf{r} &= 0, & \frac{d}{dt_U}\mathbf{p} &= -\nabla_r U\,. \end{aligned} \tag{5}$$

Both equations are integrable, and simple to integrate. The solution of $\dot{\mathbf{a}} = \mathbf{v} \times \mathbf{a}$ with constant $\mathbf{v}$ is a rotation $\mathbf{a}(t) = R(t\mathbf{v})\mathbf{a}(0)$ where the linear operator $R(t\mathbf{v})$ is the rotation about the axis $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ by the amount $t|\mathbf{v}|$. With $\mathbf{u} = t\mathbf{v}$ and $\hat{\mathbf{u}} = \mathbf{u}/|\mathbf{u}|$ we have

$$\begin{aligned} R(\mathbf{u})\mathbf{a} &= (1 - \cos(|\mathbf{u}|))(\hat{\mathbf{u}} \cdot \mathbf{a})\,\hat{\mathbf{u}} + \cos(|\mathbf{u}|)\,\mathbf{a} + \sin(|\mathbf{u}|)\,\hat{\mathbf{u}} \times \mathbf{a}\,, \\ &= \left(Id + \frac{\sin|\mathbf{u}|}{|\mathbf{u}|}(\mathbf{u}\times) + \frac{1}{2}\left(\frac{\sin|\mathbf{u}|/2}{|\mathbf{u}|/2}\right)^2 (\mathbf{u}\times)^2 \right)\mathbf{a}\,, \end{aligned}$$

where $(\mathbf{u}\times)$ is the antisymmetric matrix with entries such that left multiplication of $\mathbf{a}$ by this matrix gives the cross product $\mathbf{u} \times \mathbf{a}$. Thus the flow of T is

$$\Phi^T_\tau(\mathbf{r}, \mathbf{p}, \mathbf{a}, \mathbf{l}) = (\mathbf{r} + \tau\mathbf{p}/m, \mathbf{p}, R(\tau\mathbf{l}/\Theta_1)\mathbf{a}, \mathbf{l})\,.$$

The flow of U is

$$\Phi_\tau^U(\mathbf{r},\mathbf{p},\mathbf{a},\mathbf{l}) = (\mathbf{r},\mathbf{p}-\tau\nabla_r U,\mathbf{a},\mathbf{l}+\tau\nabla_a U\times\mathbf{a})\,.$$

Since these flows are generated by "Hamiltonians" T and U with respect to the Poisson structure they automatically preserve the Poisson structure. They are so called Poisson maps Φ, which satisfy

$$\{F\circ\Phi, G\circ\Phi\} = \{F,G\}\circ\Phi\,,$$

for arbitrary functions F, G. A symplectic map is a special case of the Poisson map for which the bracket is the canonical bracket.

A first order Poisson integrator with step size h is now given by

$$\Phi_h := \Phi_h^T\circ\Phi_h^U$$

Explicitly we find for Φ_h

$$\begin{aligned}
\Phi_h : (\mathbf{r},\mathbf{p},\mathbf{a},\mathbf{l}) &\mapsto (\mathbf{r}',\mathbf{p}',\mathbf{a}',\mathbf{l}') \\
\mathbf{p}' &= \mathbf{p} - h\nabla_r U(\mathbf{r},\mathbf{a}) \\
\mathbf{l}' &= \mathbf{l} + h\nabla_a U(\mathbf{r},\mathbf{a})\times\mathbf{a} \\
\mathbf{r}' &= \mathbf{r} + h\mathbf{p}'/m \\
\mathbf{a}' &= R(h\mathbf{l}'/\Theta_1)\mathbf{a}
\end{aligned}$$

Another first order integrator is given by the composition in the other order

$$\Phi_h^* := \Phi_h^U\circ\Phi_h^T\,.$$

and it is explicitly given by

$$\begin{aligned}
\Phi_h^* : (\mathbf{r},\mathbf{p},\mathbf{a},\mathbf{l}) &\mapsto (\mathbf{r}',\mathbf{p}',\mathbf{a}',\mathbf{l}') \\
\mathbf{r}' &= \mathbf{r} + h\mathbf{p}/m \\
\mathbf{a}' &= R(h\mathbf{l}/\Theta_1)\mathbf{a} \\
\mathbf{p}' &= \mathbf{p} - h\nabla_r U(\mathbf{r}',\mathbf{a}') \\
\mathbf{l}' &= \mathbf{l} + h\nabla_a U(\mathbf{r}',\mathbf{a}')\times\mathbf{a}'
\end{aligned}$$

Using that $R^{-1}(-t\mathbf{v}) = R(t\mathbf{v})$ it is easy to see that this is in fact the method adjoint to Φ_h, namely

$$\Phi_h^* = (\Phi_{-h})^{-1}\,.$$

Either of the two maps Φ_h and Φ_h^* can be considered as the "Standard map" for the problem of a symmetric rigid body in space.

For a long time integration a higher order method is needed. Moreover, neither Φ_h nor Φ_h^* are reversible methods, i.e. they do not satisfy

$\Phi_h \circ \Phi_{-h} = id$, namely $\Phi_h \neq \Phi_h^*$. A second order reversible integrator can be defined (see, e.g., [8]) as the composition

$$\tilde{\Phi}_h := \Phi^*_{h/2} \circ \Phi_{h/2} \,.$$

Composing the two with half steps of h and using the flow property $R(s\mathbf{v})R(t\mathbf{v}) = R((s+t)\mathbf{v})$ the second order integrator $\tilde{\Phi}_h$ explicitly is given by

$$\begin{aligned}
\tilde{\Phi}_h : (\mathbf{r}, \mathbf{p}, \mathbf{a}, \mathbf{l}) &\mapsto (\mathbf{r}'', \mathbf{p}'', \mathbf{a}'', \mathbf{l}'') \\
\mathbf{r}'' &= \mathbf{r} + \frac{h}{m}\mathbf{p}' \\
\mathbf{a}'' &= R(h\mathbf{l}'/\Theta_1)\mathbf{a} \\
\mathbf{p}'' &= \mathbf{p}' - \frac{h}{2}\nabla_r U(\mathbf{r}'', \mathbf{a}'') \\
\mathbf{l}'' &= \mathbf{l}' + \frac{h}{2}\nabla_a U(\mathbf{r}'', \mathbf{a}'') \times \mathbf{a}''
\end{aligned}$$

where $\mathbf{p}'$ and $\mathbf{l}'$ are given by $\Phi_{h/2}$, i.e.

$$\mathbf{p}' = \mathbf{p} - \frac{h}{2}\nabla_r U(\mathbf{r}, \mathbf{a}), \quad \mathbf{l}' = \mathbf{l} + \frac{h}{2}\nabla_a U(\mathbf{r}, \mathbf{a}) \times \mathbf{a} \,.$$

Since the adjoint is given by the composition of the partial flows in the other order, $\Phi_h^* = \Phi_h^U \circ \Phi_h^T$, clearly the second order reversible integrator is given by the three step composition $\Phi^U_{h/2} \circ \Phi^T_h \circ \Phi^U_{h/2}$. In the setting of splitting methods this is known as Strang scheme. For the symplectic part $(\mathbf{r}, \mathbf{p})$ this gives the classical Störmer-Verlet scheme. Hence we may say that $\tilde{\Phi}_h$ is an extension of the symplectic Störmer-Verlet scheme to a Poisson scheme for symmetric rigid bodies. A proof that this is a second order integrator can be found in [8]. Moreover, from the building blocks Φ_h and Φ_h^* also higher order integrators can be constructed, see [8] and the references therein.

Similar constructions apply to slightly more general Hamiltonians. Namely it is enough that the kinetic energy satisfies $\nabla_l T || \mathbf{l}$. However, such Hamiltonians might not have the interpretation of a symmetric rigid body in space.

For the integration of a system of N symmetric rigid bodies the same scheme $\tilde{\Phi}_h$ can be used, one simply needs to put the labels $i = 1, \ldots, N$ to all the vectors $(\mathbf{r}, \mathbf{a}, \mathbf{p}, \mathbf{l})$.

4 Symmetric Potential

Now we assume that the potential $U(\mathbf{r}, \mathbf{a})$ has additional symmetry. For the case of the Levitron this is an additional S^1 symmetry, correponding to the rotation about the spatial z-axis. For systems of symmetric rigid bodies see below. We now show that the integrators constructed here conserve these additional integrals.

The kinetic energy of the symmetric top is already manifestly rotationally symmetric. In the original system the S^1 symmetry acts by left multiplication (unlike the body symmetry). For the potential we hence assume

$$U(\mathbf{R}_z\mathbf{r}, \mathbf{R}_z\mathbf{a}) = U(\mathbf{r}, \mathbf{a}) ,$$

where $\mathbf{R}_z$ is a rotation that fixes the z-axis. Under this assumption by Noether's theorem the system has another conserved momentum

$$j_z = \langle \mathbf{r} \times \mathbf{p} + \mathbf{l}, \mathbf{e}_z \rangle .$$

which is the z-component of the total angular momentum $\mathbf{j} = \mathbf{r} \times \mathbf{p} + \mathbf{l}$.

Let us call an integrator *momentum preserving* if it not only preserves the Casimir C_1 and the body angular momentum $C_2 = L_3$ (another Casimir), but also the total spatial angular momentum about the z-axis j_z.

Proposition 1. *Φ_h is a first order momentum preserving Poisson integrator for the symmetric rigid body with symmetric potential.*

Proof. Both properties, Poisson and momentum preserving, follow from the fact that this is a splitting integrator. Each partial flow certainly is a Poisson map. Also, automatically the Casimirs are exactly preserved by a flow generated from the Poisson bracket. The only thing to check is wether both, Φ_h^U and Φ_h^T preserve the additional conserved momentum j_z. This follows from the fact that each flow already preserves j_z, i.e. $\{j_z, T\} = 0$ and $\{j_z, U\} = 0$, using the Poisson bracket defined above. The first is easily checked, for the second identity one needs the invariance of U under rotations about the z-axis, which was assumed above. □

Proposition 2. *$\tilde{\Phi}_h$ is a second order reversible momentum preserving Poisson integrator for the symmetric rigid body with symmetric potential.*

Proof. The fact that $\tilde{\Phi}_h$ is Poisson and momentum preserving follows from the fact that it is obtained from a composition of momentum preserving methods. We have directly shown that Φ_h is momentum preserving in proposition 1, here we just need the fact that the adjoint Φ_h^* has the same property. This is evident because the sign of h in Φ_h does not change momentum preservation, and since Φ_{-h} is a diffeomorphism its inverse preserves momentum as well. The composition of a method with its adjoint always gives a reversible method. The fact that it is a second order reversible method follows from the general theory, see e.g. [8]. □

Similar results can be obtained for a system of N symmetric rigid bodies, but the details of course depend on the Hamiltonian. The $\mathbb{T}^N$ body symmetryhas already been taken care of by reduction, and the result is the Poisson structure that replaces the symplectic structure. Typically the additional symmetry group for a system of rigid bodies will be $E(3)$, so that the total linear momentum $\sum \mathbf{p}_i$ and the total angular momentum $\sum \mathbf{l}_i + \mathbf{r}_i \times \mathbf{p}_i$ will be conserved. An example with this symmetry is $H = T + U$ with

$$T = T(|\mathbf{p}_i|, |\mathbf{l}_i|),$$
$$U = U(\langle \mathbf{r}_i - \mathbf{r}_j, \mathbf{r}_k - \mathbf{r}_l \rangle, \langle \mathbf{r}_i - \mathbf{r}_j, \mathbf{a}_k \rangle, \langle \mathbf{a}_i, \mathbf{a}_j \rangle)\,,$$

which is clearly invariant under $\mathbf{r}$ translations and arbitrary rotations. The arguments of T and U are obtained by letting the indices i, j, k, l in each scalar product have all possible values except $i = j$ and $k = l$. The rigid bodies can have different masses and moments of inertia.

If the Hamiltonian $H = T + U$ has a $E(3)$ symmetry and the splitting method can be applied then already T and U individually have that same symmetry. But then each partial flow preserves the momenta, and therefore the second order reversible integrator $\tilde{\Phi}_h$ also does.

A simple special case of the above method can also be applied to the Lagrange top, i.e. the symmetric rigid body with a fixed point in the symmetric field of gravity, $U(\mathbf{a}) = da_z$. It is described in the space fixed (!) frame and fixing a point in the body simply means to remove $\mathbf{r}$ and $\mathbf{p}$ and their equations.

5 The Levitron

The Levitron is a symmetric rigid body (without a fixed point) with symmetric potential. The kinetic and potential energy are given by

$$\begin{aligned} T &= \frac{\mathbf{p}^2}{2m} + \frac{\mathbf{l}^2}{2\Theta_1} \\ U &= mgz - \langle \mathbf{B}(\mathbf{r}), \mu \mathbf{a} \rangle \end{aligned}$$

In this case the equations of motion read

$$\begin{aligned} \dot{\mathbf{a}} &= \frac{1}{\Theta_1} \mathbf{l} \times \mathbf{a}, \quad & \dot{\mathbf{l}} &= \mu \mathbf{a} \times \mathbf{B}(\mathbf{r}), \\ \dot{\mathbf{r}} &= \frac{1}{m} \mathbf{p}, & \dot{\mathbf{p}} &= \mu \nabla_r \langle \mathbf{B}(\mathbf{r}), \mathbf{a} \rangle - g \mathbf{e}_z\,. \end{aligned} \tag{6}$$

The conservation of j_z follows from the rotational symmetry of the magnetic field,

$$\mathbf{B}(\mathbf{R}_z(\mathbf{r})) = \mathbf{R}_z(\mathbf{B}(\mathbf{r}))\,,$$

and the invariance of the scalar product under rotations.

The magnetic field is determined by a potential V by $\mathbf{B}(\mathbf{r}) = -\nabla V(\mathbf{r})$. Exploiting the fact that $\nabla \mathbf{B} = -\Delta V = 0$ and the symmetry which implies that $V(\mathbf{r}) = F(x^2 + y^2, z)$ the function F is determined by the field strength on the symmetry axis $\phi(z)$ by

$$F(x^2 + y^2, z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2^j j!)^2} (x^2 + y^2)^j \frac{\mathrm{d}^{2j}\phi}{\mathrm{d}z^{2j}}(z)\,,$$

extending the computation of [3] to arbitrary order. In the numerics we truncate this series at $j = 7$. Harmonicity of V implies that $4F_1 + 4(x^2 + y^2)F_{11} + F_{22} = 0$, where indices on F denote derivatives with respect to the corresponding argument. Using this relation small non-harmonicity due to truncation and numerical errors can be corrected for. We consider the simplest model for the base magnet, namely that of a uniformly magnetised disk of radius a, for which

$$\phi(z) = 2\pi \left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right).$$

The convergence of the series expansion of F is good as long as $x^2 + y^2 < a^2$. But this is not a problem, because when this inequality is violated the top is falling already.

By a scaling we can put Θ_1, m, and g equal to 1. This means to measure mass in units of m, length in units of the "dynamical width" of the top $\alpha = \sqrt{\Theta_1/m}$, and time in units of the "pendulum period" $\sqrt{\alpha/g}$. The remaining parameter is $\tilde{\mu}$ which is the ratio of the magnetic energies $\mu|\mathbf{B}|/\alpha$ and the gravitational energy $mg\alpha$. In addition, the conserved angular momentum L_3, which depends on the initial conditions in the full system, is a parameter of the reduced system.

The equilibrium point of (6) at $\mathbf{r} = (0, 0, z_s)$, $\mathbf{a} = (0, 0, 1)$, $\mathbf{p} = (0, 0, 0)$, $\mathbf{l} = (0, 0, l_z)$ is a relative equilibrium of the Levitron: the top is sitting on the spatial symmetry axis with the symmetry axis of the top aligned to the spatial symmetry axis and is rotating about that axis. In [3] we have shown under which conditions on ϕ and the parameters this relative equilibrium is linearly stable. But the Hessian (with respect to local symplectic coordinates near the equilibrium) of the Hamiltonian is not definite, and Lyapunov stability may not hold due to Arnold diffusion. In fact one can show that even in the span of the two Hessians of H and of j_z there is no positive definite matrix. Thus stability cannot be proved using energy-momentum methods, and the long term fate of orbits starting near the equilibrium point is unclear.

6 The Escape Time Diagram

The integration method $\tilde{\Phi}_h$ can be used to calculate the time that is needed to escape a neighborhood of the relative equilibrium on a grid of initial conditions. Escape is defined by the condition that the centre of mass leaves a spherical neighbourhood of the equilibrium point that has radius either the radius of the base of the magnet, or the height of the relative equilibrium above the base, whichever is smaller.

After reduction the Levitron has still 5 degrees of freedom. There is one remaining constant of motion j_z, which can, however, not be reduced without introducing a singularity at the relative equilibirum. It is not possible to study the full 10 dimensional neighborhood of the equilibrium. We chose to consider certain two dimensional sections through phase space.

The initial conditions are $\mathbf{r} = (x, 0, z_s + z)$, $\mathbf{a} = (0, 0, 1)$, $\mathbf{p} = (0, 0, 0)$, $\mathbf{l} = (0, 0, \sigma\Theta_3)$, where z_s is the z-coordinate of the linearly stable equilibrium of the Levitron, and (x, z) are small displacements. The orbit of this initial condition either escapes after time $t < T_{\max}$ or not. This defines a map from points in the plane (x, z) into the real numbers, which is color coded in the escape time diagram, see the figures. The set of initial conditions that do not leave up to time $T_{\max}$ is an approximation to the stable region. Our numerical experiments show that the boundary of the stable region has a complicated structure. One might expect that in the limit $T_{\max} \to \infty$ it develops a more and more fractal structure.

The escape time in the (x, z) plane is large near the equilibrium. If the spin rate σ is chosen so that the equilibrium is linearly stable then always $T_{\max}$ is reached sufficiently close to the equilibrium. This follows from Nekhoroshev estimates, assuming the steepness of the system, see [5]. Since the estimates give exponentially long times this is numerically not accessible. Nevertheless by choosing e.g. $T_{\max} = 20$ in Figure 1a the complicated boundary of the stable set becomes apparent. The distance at which the instability becomes visible is too large for Nekhoroshev estimates to hold. The complicated structure of the boundary is illustrated in Figure 1b. An enlargement in the space of initial conditions is made, together with increasing the escape time to $T_{\max} = 100$. Further enlargements take a prohibitively long time to compute.

7 Acknowledgement

The author would like to thank Sebastian Kubiesa for doing the first numerical experiments computing the stability boundary in his final

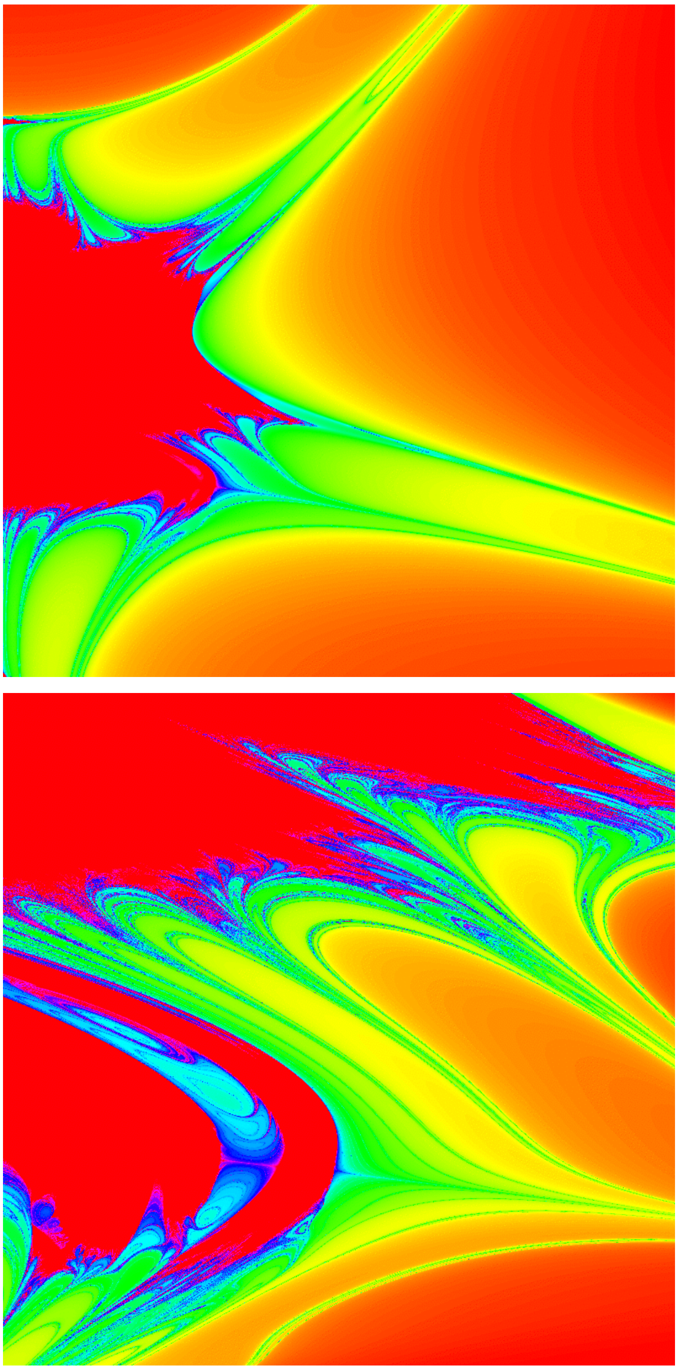

Figure 1: Escape time diagram for the Levitron with parameters $m = 0.02, a = 0.05, \mu = -0.000095, \Theta_1 = m(0.3a)^2/4$ so that $z_s = 0.0313$. The initial conditions are as explained in the text, with stable spin rate $\sigma = 20$. a) In the upper figure the window is $[0, 0.012] \times [z_s - 0.01, z_s + 0.01]$, and $T_{\max} = 20$ has been used, together with $h = 0.002$. b) In the lower figure an enlargement is shown with size $[0.0028, 0.002] \times [z_s - 0.0035 - 0.002, z_s - 0.0035 + 0.002]$, and $T_{\max} = 100$

year project. This research was supported by the European Research Training Network *Mechanics and Symmetry in Europe* (MASIE), HPRN-CT-2000-00113. The hospitality of the Bernoulli Centre at EPF Lausanne in Summer 2004 during which this work was completed is gratefully acknowledged.

References

[1] E. Barth and B. Leimkuhler. Symplectic methods for conservative multibody systems. In *Integration algorithms and classical mechanics*, volume 10 of *Fields Inst. Commun.*, pages 25–43. Amer. Math. Soc., 1996.

[2] M. V. Berry. The levitron: an adiabatic trap for spins. *Proc. Royal Soc. London, Series A*, 452:1207–1220, 1996.

[3] H. R. Dullin and R. W. Easton. Stability of Levitrons. *Physica D*, 126:1–17, 1999.

[4] H. R. Dullin and R. W. Easton. Stability of Levitrons. *Z. Angew. Math. Mech.*, 79:S167–S170, 1999. suppl. 1: proceedings of the GAMM 98.

[5] H. R. Dullin and F. Fassò. An algorithm for detecting directional quasi-convexity. *BIT numerical mathematics*, (in print), 2004.

[6] A. Dullweber, B. Leimkuhler, and R. McLachlan. Symplectic splitting methods for rigid body molecular dynamics *J. Chem. Phys.*, 107:5840-5851, 1997

[7] US patent 4,382,245 Roy M. Harrigan, 1983.

[8] E. Hairer, C. Lubich, and G. Wanner. *Geometric numerical integration.* Springer, Berlin, 2002.

[9] R. I. McLachlan and G. R. Quispel. Splitting methods. *Acta Numer.*, 11:341–434, 2002.

[10] S. Reich. Symplectic integrators for systems of rigid bodies. In *Integration algorithms and classical mechanics*, volume 10 of *Fields Inst. Commun.*, pages 181–191. Amer. Math. Soc., 1996.

[11] J. M. Sanz-Serna and M. P. Calvo. *Numerical Hamiltonian problems.* Chapman & Hall, London, 1994.

[12] M. D. Simon, L. O. Heflinger, and S. L. Ridgway. Spin stabilized magnetic levitation. *Am. J. Phys.*, 65:286, 1997.